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Difference Equations in Statistical Mechanics. I. Cluster Statistics Models

V. Privman[1] and N. M. Švrakić[1]

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A review and some new results are presented for several cluster statistics models, solutions of which can be reduced to difference equations. Mathematical techniques suitable for solving these equations are surveyed.

KEY WORDS: Lattice animals; directed models; stacking models; generating functions; critical phenomena; continued fractions.

1. INTRODUCTION

Cluster statistics models, such as random walks, lattice animals (connected clusters), surfaces, and solid-on-solid strings and sheets, serve as prototype systems with "geometric" phase transitions. Typically, such models involve a set of N connected points (sites) or links (bonds) on a d-dimensional lattice. Geometric constraints such as self-avoidance, directedness, or compactness are imposed. Universal features of the large-N behavior of the cluster number (entropy), sizes, and shape are then investigated. However, *exact* results can be obtained only in a limited number of cases, especially when both the universal critical and the *global* behaviors are concerned. For example, the isotropic percolation models have proved to be notoriously resistant to analytic treatment. On the other hand, many variations of the non-self-avoiding random walk models can be solved exactly.

In this paper we review a class of compact $2d$ cluster models that in many cases can be solved exactly for the partition functions that generate the cluster numbers. We also report various new results. The partition functions are obtained as solutions of difference equations. The appropriate mathematics are sometimes rather complicated, involving objects such as

[1] Department of Physics, Clarkson University, Potsdam, New York 13676.

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q-series, continued fractions, etc. We survey some of the mathematical methods, including several results developed quite recently. An accompanying article[1] reviews recent advances in the applications of difference equations to studies of solid-on-solid models of surface critical phenomena in two dimensions.

In Section 2, we consider a model of stackings of squares at a line wall.[2] It can be easily solved[3] and will serve to illustrate some of the general features of the compact lattice animal models. A model of stackings of circles[2] in a triangular lattice array at a line wall is considered in Section 3. The solution of this model[4] involves q-series and fully illustrates the mathematical complexity of some of the compact animal models. Two methods that are useful in solving the compact animal models, namely the use of continued fractions and the generating function approach, are outlined in Sections 4 and 5, respectively. In Sections 6 and 7, we discuss the so-called[2] "one-tooth" versions of the square and circle stackings,[2,5] and mention results for a related model[2,6] of filling a corner by squares. We then turn to the compact directed animal models on partially and fully directed square lattices (Sections 8 and 9, respectively).[7–10,4] Additional mathematical aspects, beyond the conventional methods, are summarized in Section 10. Finally, some open problems, $3d$ model[10,11] results, etc., are mentioned in Section 11, which is devoted to discussion and summary.

2. STACKINGS OF SQUARES AT A LINE WALL

Following Temperley,[2] we consider a $2d$ "castle wall" built up from N squares, as shown in Fig. 1. The base row of the cluster must be continuous. Higher rows can have gaps. However, each column must be continuous "self-supporting." One can easily calculate[3] the total number c_N of different N-site clusters, i.e., the number of possible arrangements of N squares consistent with the above rules. In order to avoid counting

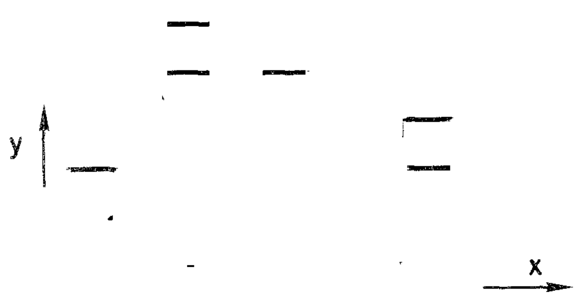

Fig. 1. Stacking of squares according to the rules of a model considered in Section 2.

clusters that differ only by overall translations, we pin the lowest leftmost square with its center at $(x, y) = (0, 0)$ (see Fig. 1).

Let $c_{N,k}$ denote the number of different clusters with exactly k squares in the leftmost $x = 0$ column ($k = 2$ in Fig. 1). Define the restricted partition functions

$$F_k(z) = \sum_{N=k}^{\infty} c_{N,k} z^{N-k}, \qquad k \geqslant 1 \tag{2.1}$$

These functions satisfy the recursion relations

$$F_k(z) = 1 + \sum_{m=1}^{\infty} z^m F_m(z) \tag{2.2}$$

where each term specifies one possible configuration of the next column centered at $x = 1$. The term 1 corresponds to no second column (the $k = N$ cluster). Each $m > 0$ term in (2.2) sums up all configurations with exactly m squares at $x = 1$, with $F_m(z)$ accounting for the distribution of squares in all the $x > 1$ columns.

The *total* partition function $G(z)$ for the cluster numbers c_N is given by

$$G(z) \equiv \sum_{N=1}^{\infty} c_N z^N = \sum_{k=1}^{\infty} z^k F_k(z) \tag{2.3}$$

where the last step is self-explanatory [see (2.1)]. Inspection of (2.2) gives

$$F_k(z) = 1 + G(z) \qquad \text{for all } k \tag{2.4}$$

Finally, substitution in (2.2) yields

$$1 + G = 1 + (1 + G) \sum_{m=1}^{\infty} z^m \tag{2.5}$$

$$G(z) = \frac{z}{1 - 2z} \tag{2.6}$$

The Taylor series coefficients of this function are the desired cluster numbers c_N [see (2.3)]. Thus,

$$c_N = 2^{N-1} \tag{2.7}$$

for the square-stacking model. The general large-N form, applicable to most lattice animal models, is

$$c_N \approx A N^{-\theta} \lambda^N \tag{2.8}$$

Here A and λ are model-dependent, while the "critical exponent" θ is universal for large classes of models that differ by the details of the "microscopic" connectivity rules, lattice structure, etc., but share global "macroscopic" features such as directedness, compactness, and dimensionality of space. Here we have $\theta = 0$.

As usual, the large-N behavior of the Taylor coefficients c_N of the partition function is controlled by the singularity nearest to the origin on the real $z > 0$ axis (since all $c_N > 0$). In the present case, the singularity is a simple pole at $z_c = 1/2$.

3. STACKINGS OF CIRCLES AT A LINE WALL

In this section we consider stackings of circles[2] at a line wall as illustrated by the open circles in Fig. 2: N circles are positioned in such a way that the base row is continuous. The higher rows can have gaps; however, each circle must be "supported" by having both lower-y neighbors occupied. The centers then follow the pattern of the triangular lattice with spacing equal to the circle diameter.

In order to solve the model,[4] we extend the allowed configurations to include additional $k-1$ base circles along a lattice direction forming $60°$ with the negative x axis. The case $k = 3$ is illustrated in Fig. 2. The $k-1 = 2$ filled circles are part of the base. Together with the open circles they can "support" additional circles (filled circles in Fig. 2).

Let $c_{N,k}$ denote the number of distinct N-circle clusters with exactly k circles in the $60°$ base (counting the origin circle, which also belongs to the horizontal base at $y = 0$, the length of which is not restricted). The restric-

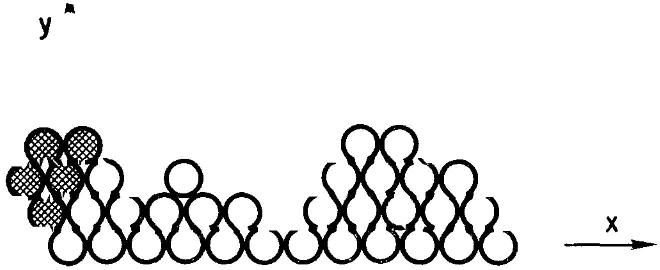

Fig. 2. Compact self-supporting stacking at the line wall (open circles). Filled circles illustrate the two additional 60° base circles (see Section 3) and the circles supported by the extended base.

ted partition functions $F_k(z)$ defined as in (2.1) satisfy the recursion relations

$$F_k(z) = 1 + \sum_{m=1}^{k+1} z^m F_m(z), \qquad k \geqslant 1 \tag{3.1}$$

As in Section 2, the terms on the right sum up configurations with different number m of circles in the 60° row next to the base 60° row. Note that by the stacking rules, m cannot exceed $k+1$. Replacing (3.1) by the first difference, we get a second-order difference equation

$$F_{k+1}(z) - F_k(z) = z^{k+2} F_{k+2}(z), \qquad k \geqslant 1 \tag{3.2}$$

with the boundary condition ($k = 1$)

$$F_1(z) = 1 + zF_1(z) + z^2 F_2(z) \tag{3.3}$$

Note that the partition function for the original circle-stacking problem, defined by the left relation in (2.3), is given here by

$$G(z) = zF_1(z) \tag{3.4}$$

The general solution of (3.2) can be represented as

$$A(z)\,\phi_k(z) + B(z)\,\Phi_k(z) \tag{3.5}$$

where[4]

$$\phi_k(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n(n+k+1)}}{q_n} \tag{3.6}$$

with $q_0 \equiv 1$ and

$$q_n \equiv \prod_{j=1}^{n} (1 - z^j) \qquad \text{for} \quad n \geqslant 1 \tag{3.7}$$

represents the "physical" or regular at $z = 0$ (for $k \geqslant -2$) solution. One can show[4] that the second linearly independent solution $\Phi_k(z)$ is power-law-singular at $z = 0$ for sufficiently large k. Furthermore, in the mathematical nomenclature,[12] $\phi_k(z)$ is the *minimal* solution, in that

$$\lim_{k \to \infty} [\phi_k(z)/\Phi_k(z)] = 0 \tag{3.8}$$

for all z values of "physical" interest, i.e., for $0 < z < 1$. More detailed discussion of the minimal solution concept is given in an accompanying article.[1] Methods of solving the second-order difference equations of the type (3.2) are discussed in Sections 4, 5, and 10.

The boundary condition (3.3) can be satisfied with $B(z) \equiv 0$ and

$$A(z) = [(1-z)\,\phi_1(z) - z^2\phi_2(z)]^{-1} \tag{3.9}$$

in (3.5). Thus, the partition function for the circle-stacking problem, relation (3.4), reduces to

$$G(z) = \frac{z}{1 - z - z^2\phi_2(z)/\phi_1(z)} \tag{3.10}$$

The q-series $\phi_k(z)$ are analytic for $|z| < 1$, with a natural boundary *at* the unit circle. However, the nearest-to-the-origin singularity of $G(z)$ is a simple pole at the first zero of the denominator of (3.10), at

$$z_c = \lambda^{-1} = 0.576148769... < 1 \tag{3.11}$$

Thus, we have $\theta = 0$ for the universal exponent in (2.8). Further details can be found in ref. 4.

4. CONTINUED FRACTION TECHNIQUES

A standard mathematical approach to linear *second-order* difference equations[12,13] utilizes continued fractions to calculate the minimal solution [see (3.8)]. We illustrate the technique for the circle-stacking problem of Section 3.

The difference equation (3.2) is reformulated in terms of the ratios

$$R_k(z) = F_{k+1}(z)/F_k(z) \tag{4.1}$$

as

$$1 - R_k^{-1}(z) = z^{k+2}R_{k+1} \tag{4.2}$$

$$R_k = \frac{1}{1 - z^{k+2}R_{k+1}} \tag{4.3}$$

This relation can be iterated to generate a "backward" continued fraction representation

$$R_k(z) = \cfrac{1}{1 - \cfrac{z^{k+2}}{1 - \cfrac{z^{k+3}}{1 - \cfrac{z^{k+4}}{1 - \cdots}}}} \tag{4.4}$$

The Pincherle theorem[14] then ensures a one-to-one correspondence between the convergence of the continued fractions (4.4) and the existence of the minimal solution given by

$$\phi_k(z)=\phi_0(z)\prod_{j=0}^{k-1} R_j(z), \qquad k\geqslant 1 \tag{4.5}$$

[Note that $\phi_k(z)$ is defined up to an arbitrary z-dependent coefficient $\phi_0(z)$, since (3.2) is linear.]

For the "physical" partition function (3.10), we could use the continued fraction representation obtained by replacing ϕ_2/ϕ_1 in (3.10) by

$$\frac{\phi_2(z)}{\phi_1(z)}=\cfrac{1}{1-\cfrac{z^3}{1-\cfrac{z^4}{1-\cfrac{z^5}{1-\cdots}}}} \tag{4.6}$$

This form can be used to reproduce all the conclusions on the analytic structure of $G(z)$ mentioned in Section 3.[4] The continued fraction representation is also not inferior to the infinite-sum forms such as (3.6) for numerical computation purposes.[13] However, the infinite-sum representations are more familiar. They can be obtained in many cases by utilizing mathematical results building on the classical work of Ramanujan.[15–17] For the identification of R_k of (4.4) with ϕ_{k+1}/ϕ_k of (3.6) one can use a result given, e.g., in ref. 15 (p. 25, third unnumbered equation).

For physical applications and, in particular, to analyze the complex-plane singularities, an *infinite-product* representation of a partition function would be valuable. We are not aware of any mathematical results appropriate for $G(z)$ here. However, infinite-product forms have been utilized in other applications of the q-series in physics.[18]

We will see in Sections 6–10 that in some cases both the minimal and some of the other linearly independent solutions are physically relevant. When one solution (minimal) is available, the order of the difference equation can be reduced by one, by standard methods.[19]

5. GENERATING FUNCTION METHOD

This approach[19] consists of considering the generating function

$$P(z,t)=\sum_{k=1}^{\infty} F_k(z)\,t^{k-1} \tag{5.1}$$

When the difference equation is multiplied by t^{k-1} and summed over k, one sometimes ends up with an equation for $P(z, t)$. Specifically, for difference equations with constant (k-independent) coefficients, algebraic equations for P are obtained. Equations with coefficients involving integral powers of k yield *differential equations* for P; this case is important in the solid-on-solid model studies.[1] However, difference equations with exponential-in-k coefficients, such as z^k, lead to functional equations for $P(z, t)$ that are rather difficult to solve in general. We will consider a few examples in the following sections.

The generating function method and related techniques,[19] e.g., Laplace's method, complement the continued fraction approach. However, they can also be used for equations of order higher than second, and in some cases are advantageous even for second-order difference equations.[1]

As already mentioned, for compact cluster statistics models considered here, one typically obtains a functional equation for $P(z, t)$ of the form

$$P(z, t) = a(z, t) + b(z, t)\, P(z, tz) \tag{5.2}$$

Mathematical literature on equations of this sort is limited.[20-22] When the resulting series is well defined, one can use a solution obtained by iterating (5.2) an infinite number of times,

$$P(z, t) = a(z, t) + \sum_{n=1}^{\infty} \left[a(z, tz^n) \prod_{m=0}^{n-1} b(z, tz^m) \right] \tag{5.3}$$

The solution of (5.2) is linear in a in the sense that for

$$a(z, t) = \alpha_1(z)\, a_1(z, t) + \alpha_2(z)\, a_2(z, t) \tag{5.4}$$

one has

$$P_a = \alpha_1(z)\, P_{a_1} + \alpha_2(z)\, P_{a_2} \tag{5.5}$$

as can be seen explicitly for (5.3).

For the circle stacking problem considered in Sections 3 and 4, the appropriate equation (5.2) has

$$a(z, t) = \frac{F_1(z) - z^2 F_2(z)}{1-t} - \frac{zF_1(z)}{t(1-t)} \tag{5.6}$$

$$b(z, t) = \frac{z}{t(1-t)} \tag{5.7}$$

Each solution is a linear combination of the type (5.5), with $a_1 = (1-t)^{-1}$, $a_2 = [t(1-t)]^{-1}$. The coefficients $\alpha(z)$ involve two unknown functions

$F_{1,2}(z)$. One relation between the coefficients is provided by the boundary condition (3.3). Another condition must therefore result from the "analyticity" requirement on $P(z, t)$ near $t=0$.

However, expansion (5.3) is ill-defined in this case. The generating function technique can be applied to this problem if we consider modified functions

$$f_k(z) = z^{k(k+1)/2} F_k(z) \tag{5.8}$$

satisfying

$$f_{k+1}(z) - f_{k+2}(z) = z^{k+1} f_k(z) \tag{5.9}$$

After a tedious calculation utilizing (5.3), one ends up with an expression for the partition function $G(z)$. However, it is much more complicated than (3.10) with (3.6) or (4.6). We omit the details of this calculation.

6. SINGLE PYRAMID STACKING OF SQUARES

In this section we consider a "one-tooth"[2] stacking of squares in a pyramidlike shape illustrated in Fig. 3. The model is similar to that of Section 2, but with the additional requirement that each row is continuous. The quantity of interest is the total number c_N of pyramids that can be built from N squares. Let $c_{N,k}$ denote the number of distinct N-square pyramids with exactly k squares in the base row. We define the restricted partition functions by (2.1). The appropriate recursion relations are[2]

$$F_k = 1 + \sum_{m=1}^{k} (k-m+1)\, z^m F_m \tag{6.1}$$

The important change here, as compared to the recursions considered in Sections 2 and 3, is the factor $(k-m+1)$ accounting for the number of

Fig. 3. Pyramid stacking of squares, defined in Section 6.

ways in which the second row of $m \leqslant k$ squares can be positioned. The $k=1, 2$ relations provide the boundary conditions, which can be simplified to

$$F_1(z)=\frac{1}{1-z}, \qquad F_2(z)=\frac{1}{(1-z)^2} \tag{6.2}$$

For higher k relations, we form second differences to obtain

$$F_{k+2}-2F_{k+1}+F_k=z^{k+2}F_{k+2} \tag{6.3}$$

Examination[2] of (6.1) suggests that all F_k are rational functions of the form

$$F_k(z)=f_k(z)/q_k(z) \tag{6.4}$$

[see (3.7)], where f_k are polynomials. Finally, note that the total partition function for the single-pyramid problem is given by

$$G(z)=P(z, z) \tag{6.5}$$

[see (2.3), (5.1)].

Let us apply the generating function method to (6.2)–(6.3). Proceeding along the lines of Section 5, we get

$$P(z, t)=\frac{(1-z-2t)\,F_1(z)+t(1-z^2)\,F_2(z)}{(1-t)^2}+\frac{z}{(1-t)^2}P(z, tz) \tag{6.6}$$

Since there are two boundary conditions, we expect that *both* linearly independent solutions are physically acceptable. In this case $F_{1,2}$ are known explicitly. Thus, (6.6) reduces to (5.2) with

$$a=\frac{1}{1-t}, \qquad b=\frac{z}{(1-t)^2} \tag{6.7}$$

By using (5.3), $P(z, t)$ is obtained,

$$P(z, t)=\sum_{n=0}^{\infty}\frac{z^n(1-tz^n)}{[\prod_{m=0}^{n}(1-tz^m)]^2} \tag{6.8}$$

For $t=z$, this is identical with the result obtained by Temperley[2] for $G(z)$ [see (6.5)] by a different method,

$$G(z)=\sum_{n=0}^{\infty}\frac{z^n(1-z^{n+1})}{q_{n+1}^2} \tag{6.9}$$

This function has an essential singularity at $z=1$, analysis of which yields the large-N cluster numbers as[5]

$$c_N \approx AN^{-5/4}\mu^{\sqrt{N}} \tag{6.10}$$

where A and μ are constants. This result is different from the generic lattice animal form (2.8). It is interesting to mention in this connection a model of filling a corner by squares[2,6] obtained by imposing additional constraint in the one-pyramid square packing model, namely that the leftmost column must not be shorter than any other column in the cluster. The resulting model is equivalent[2,6] to enumeration of nonincreasing partitions of N. Thus, detailed results are available,[2,6,23] specifically

$$c_N \approx AN^{-1}\mu^{\sqrt{N}} \tag{6.11}$$

(with different A and μ). The exponent of the power-law factors in (6.10)–(6.11) is not universal, unlike θ in (2.8).

Turning back to the one-pyramid packings of squares, we derive an explicit form for the restricted partition functions $F_k(z)$ by expanding (6.8) in powers of t. We use the identity[23]

$$\left[\prod_{m=0}^{n-1}(1-tz^m)\right]^{-1} = \sum_{j=0}^{\infty}\frac{q_{n+j-1}(z)\,t^j}{q_{n-1}(z)\,q_j(z)} \tag{6.12}$$

to get

$$F_k(z) = \sum_{n=0}^{\infty}\frac{z^n}{q_{n-1}^2}\left(\tau_{k,n} - z^n\tau_{k-1,n}\right) \tag{6.13}$$

where

$$\tau_{k,n} \equiv \sum_{j=0}^{k}\frac{q_{n+j-1}(z)\,q_{n+k-j-1}(z)}{q_j(z)\,q_{k-j}(z)} \tag{6.14}$$

This double-sum representation is not very illuminating. (See further below.)

The continued fraction method can be used to derive the form of the minimal solution of (6.3). We will only quote some results here. Continued fractions of the type appropriate for (6.3) were analyzed by Ramanujan.[17] As a result, an infinite series representation is available,

$$\phi_k(z) = \sum_{n=0}^{\infty}\frac{z^{n(n+k+1)}}{q_n^2} \tag{6.15}$$

However, for this problem the other linearly independent solution Φ_k is also physically admissible. By reducing the order of the equation using a known solution,[19] we get an extremely complicated result.

However, a relatively simple representation for $\Phi_k(z)$ can be obtained[24] by the method described in Section 10. Thus,

$$\Phi_k(z) = k + \sum_{n=1}^{\infty} \frac{z^{n(n+k+1)}}{q_n^2} [k + s_n(z)] \tag{6.16}$$

where

$$s_n(z) \equiv 2 \sum_{j=1}^{n} \frac{1}{1 - z^j} \qquad \text{for} \quad n \geqslant 1 \tag{6.17}$$

By imposing the boundary conditions, we get

$$F_k = \frac{(1-z)(\phi_k \Phi_2 - \phi_2 \Phi_k) - (\phi_k \Phi_1 - \phi_1 \Phi_k)}{(1-z)^2 (\phi_1 \Phi_2 - \phi_2 \Phi_1)} \tag{6.18}$$

Another solution of (6.3), $\psi_k(z)$, linearly independent of $\phi_k(z)$, is known in the theory of q-ultraspherical polynomials,[21]

$$\psi_k(z) = \sum_{m=0}^{k} \frac{1}{q_m q_{k-m}} \tag{6.19}$$

This can be used in place of Φ_k in (6.18).

7. SINGLE PYRAMID STACKING OF CIRCLES

Pyramid-shape or "one tooth" stacking of circles[2,5] is illustrated in Fig. 4. The stacking rules are identical to those of Section 3 (Fig. 2), but now each row must be continuous (no gaps). Let $c_{N,k}$ denote the number of distinct N-circle clusters with k circles in the base. Note that $c_N \equiv$

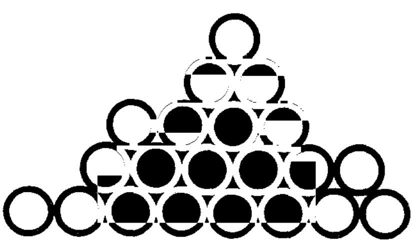

Fig. 4. Pyramid stacking of circles, defined in Section 7.

$\sum_{k=\kappa}^{N} c_{N,k}$, where κ is the smallest integer greater than or equal to $[(8N+1)^{1/2}-1]/2$. The restricted generating functions satisfy

$$F_k(z)=1+\sum_{m=1}^{k-1}(k-m)\,z^m F_m(z), \qquad k\geqslant 2 \tag{7.1}$$

with $F_1(z)=1$. By forming the second difference, this is reduced to

$$F_{k+2}-2F_{k+1}+F_k=z^{k+1}F_{k+1}, \qquad k\geqslant 1 \tag{7.2}$$

with the boundary conditions

$$F_1=1, \qquad F_2=1+z \tag{7.3}$$

Examination of (7.1) leads to the conclusion that all $F_k(z)$ are polynomials of degree $k(k-1)/2$. The structure of this problem is quite similar to that of the pyramid-of-squares packings considered in Section 6. Therefore, we only list some central results here.

The generating function method can be invoked for the pyramid circle packings. As in Section 6, the boundary conditions (7.3) are used to yield

$$a(z,t)=\frac{1}{1-t}, \qquad b(z,t)=\frac{zt}{(1-t)^2} \tag{7.4}$$

Thus, (5.3) takes the form

$$P(z,t)=\sum_{n=0}^{\infty}\frac{t^n z^{n(n+1)/2}(1-tz^n)}{[\prod_{m=0}^{n}(1-tz^m)]^2} \tag{7.5}$$

The total partition function is given by

$$G(z)=P(z,z)=\sum_{n=1}^{\infty}\frac{z^{(n-1)(n+2)/2}(1-z^n)}{q_n^2(z)} \tag{7.6}$$

identical with the result of Auluck[5] obtained by a different method. It has an essential singularity at $z_c=1$, analysis of which yields

$$c_N\sim\mu^{\sqrt{N}} \tag{7.7}$$

(Unfortunately, the form of the power-law prefactor here is not known.)

Difference equation (7.2) can be also analyzed by continued fraction techniques: we only quote the form of the minimal solution,

$$\phi_k(z)=\sum_{n=0}^{\infty}\frac{z^{n(n+2k+1)/2}}{q_n^2(z)} \tag{7.8}$$

This infinite series form of $\phi_k(z)$ follows from yet another of the Ramanujan's results.[17] In fact, both solutions of (7.2) are physically acceptable. The second solution can be found[24] by the method of Section 10,

$$\Phi_k(z)=k+\sum_{n=1}^{\infty}\frac{z^{n(n+2k+1)/2}}{q_n^2}(k-n+s_n) \tag{7.9}$$

The boundary conditions then imply

$$F_k=\frac{(1+z)(\phi_1\Phi_k-\phi_k\Phi_1)-(\phi_2\Phi_k-\phi_k\Phi_2)}{\phi_1\Phi_2-\phi_2\Phi_1} \tag{7.10}$$

8. PARTIALLY DIRECTED COMPACT LATTICE ANIMALS

In this section we consider the partially directed compact lattice animal model[7,9,10] illustrated in Fig. 5. The model is most easily described as having N squares positioned in (continuous) columns. The neighboring columns must touch by at least one square. A formulation according to directed square lattice animal rules is also possible.[9]

Let k denote the number of squares in the leftmost "root" column and $c_{N,k}$ be the number of distinct N-square k-root clusters. Then the restricted partition functions (2.1) satisfy

$$F_k(z)=1+\sum_{m=1}^{\infty}(k+m-1)\,z^m F_m(z) \tag{8.1}$$

By forming the second difference, we get

$$F_{k+2}-2F_{k+1}+F_k=0, \qquad k\geqslant 1 \tag{8.2}$$

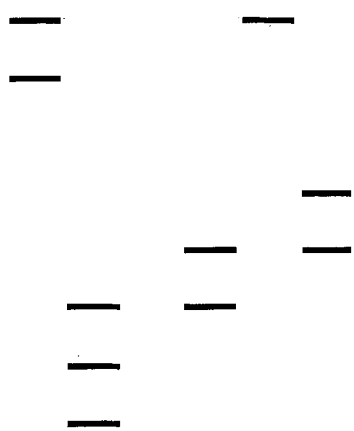

Fig. 5. A partially directed compact lattice animal (Section 8).

with the boundary conditions $[k=1, 2$ in $(8.1)]$

$$F_1 = 1 + \sum_{m=1}^{\infty} m z^m F_m \tag{8.3}$$

$$F_2 = 1 + \sum_{m=1}^{\infty} (m+1) z^m F_m \tag{8.4}$$

This problem is interesting in that (8.2) has constant coefficients. Thus, it can be solved in full detail. Specifically, the minimal solution is simply $\phi_k(z) = 1$, while the other linearly independent solution is $\Phi_k(z) = k$. Both solutions are physically acceptable. We have

$$F_k(z) = A(z) + B(z)k \tag{8.5}$$

The coefficient functions A and B are determined by (8.3)–(8.4). One gets[9]

$$F_k(z) = \frac{kz(1-z)^3 + (1-3z+z^2)(1-z)^2}{1-5z+7z^2-4z^3} \tag{8.6}$$

The total partition function is given by

$$G(z) = \sum_{k=1}^{\infty} z^k F_k(z) = \frac{z(1-z)^3}{1-5z+7z^2-4z^3} \tag{8.7}$$

All the partition functions (8.6)–(8.7) have a simple pole singularity at $z_c = \lambda^{-1}$, where

$$\lambda = 3.20556943... \tag{8.8}$$

or

$$\lambda = 12/[(6\sqrt{177}-71)^{1/3} - (6\sqrt{177}+71)^{1/3} + 7] \tag{8.9}$$

Thus, (2.8) applies, with $\theta = 0$.

9. FULLY DIRECTED COMPACT LATTICE ANIMALS

Although the model considered in this section can be defined according to the square lattice directed animal rules,[8,9] it can also be described as stackings of circles: see Fig. 6. Continuous (no horizontal gaps) rows of circles are put on top of each other with the requirement that each circle not in the base is supported by having at least *one* of its lower neighbors present. (Thus, the difference with the pyramid stackings of circles is that there "supported" meant both lower neighbors occupied.)

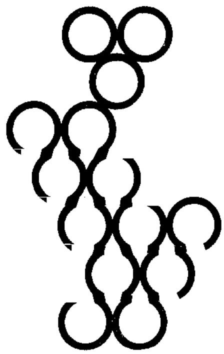

Fig. 6. A fully directed compact lattice animal (Section 9).

As usual, we consider the restricted partition functions $F_k(z)$ for the numbers $c_{N,k}$ of N-circle, k-base clusters, satisfying[8,9]

$$F_k(z) = 1 + \sum_{m=1}^{k+1} (k-m+2)\, z^m F_m(z) \tag{9.1}$$

Except for the two boundary conditions

$$F_1 = 1 + 2zF_1 + z^2 F_2 \tag{9.2}$$

$$F_2 = 1 + 3zF_1 + 2z^2 F_2 + z^3 F_3 \tag{9.3}$$

the recursions can be reduced to[9]

$$F_{k+2}(z) - 2F_{k+1}(z) + F_k(z) = z^{k+3} F_{k+3}(z) \tag{9.4}$$

This is a third-order equation. Since there are two boundary conditions, one anticipates two physically "regular" and one "irregular" solutions. Various studies[8,9,24] have found a simple pole singularity in the partition functions [i.e., $\theta = 0$ in (2.8)] at $z_c = \lambda^{-1}$, with

$$\lambda = 2.661857944... \tag{9.5}$$

The exact solution of this model has been achieved[24] along the following lines. Note that all the difference equations encountered in the cluster statistics models have discrete first or second derivatives on their left-hand sides. This is related to the fact that the multiplicity factors, such as $(k-m+2)$ in (9.1) are, respectively, constants or linear functions in their k dependence. By inspecting the minimal solutions obtained for the second-order difference equations in the preceding sections, we can *guess* one solution for

$$v_{k+1}(z)-v_k(z)=z^{k+m}v_{k+l}(z), \qquad l\geqslant 0 \tag{9.6}$$

It is

$$v_k(z)=\sum_{n=0}^{\infty}\frac{(-1)^n}{q_n}z^{n[l(n-1)+2(k+m)]/2} \tag{9.7}$$

Similarly,

$$u_{k+2}(z)-2u_{k+1}(z)+u_k(z)=z^{k+m}u_{k+l}(z), \qquad l\geqslant 0 \tag{9.8}$$

is solved by

$$u_k(z)=\sum_{n=0}^{\infty}q_n^{-2}z^{n[l(n-1)+2(k+m)]/2} \tag{9.9}$$

Thus, we have one solution for (9.4),[4]

$$\sum_{n=0}^{\infty}\frac{z^{n(3n+2k+3)/2}}{q_n^2} \tag{9.10}$$

regular at $z=0$ for $k\geqslant -3$. Formally, one can then reduce the order of the difference equation and apply the continued fraction method to derive the second regular solution. However, the resulting expressions turn out to be extremely complicated. The generating function method is also not useful because (5.3) is ill-defined.

The second "physical" solution, $\Phi_k(z)$, has been obtained[24] by a different method, described in the next section. We only quote the result for Φ_k here (see ref. 24 for further details),

$$\Phi_k=k+\sum_{n=1}^{\infty}\frac{z^{n(3n+2k+3)/2}}{q_n^2}(k+n+s_n) \tag{9.11}$$

10. LARGE-*k* ASYMPTOTIC BEHAVIOR OF THE "PHYSICAL" SOLUTIONS. RELATED MATHEMATICAL DEVELOPMENTS

The nonautonomous difference equations encountered in our review of compact cluster models in Section 2–9 where always of the form (9.6) or (9.8). Depending on the value of the nonnegative integer l, these equations may have several solutions. However, the physically acceptable solutions must be regular for small z. Furthermore, the series

$$\sum_{k=1}^{\infty}z^kF_k(z) \tag{10.1}$$

in (2.3), etc., must converge in the physically relevant part of the range $0 \leqslant z < 1$, since (10.1) always represents some sort of partition function.

Thus, for large k, the right sides of (9.6) and (9.8) asymptotically vanish for the "physical" solutions. It follows that there is exactly one such solution, (9.7), of (9.6), i.e., $\phi_k(z) \equiv v_k(z)$. It has a finite limit $v_\infty(z) \equiv 1$ as $k \to \infty$ for fixed $0 \leqslant z < 1$. Difference equation (9.8), however, has *two* "physical" solutions, $\phi_k(z) \equiv u_k(z)$ of (9.9), and $\Phi_k(z)$, with large-k behaviors

$$u_k(z) \to 1 \tag{10.2}$$

and

$$\Phi_k(z) \sim \rho_\infty(z) k, \qquad \text{for} \quad 0 \leqslant z < 1 \tag{10.3}$$

It is interesting to recall that we always had exactly the right number of boundary conditions in Sections 3, 6, 7, and 9, requiring one solution of (9.6), but two solutions of (9.8).

The asymptotically linear-in-k solution of (9.8) can be obtained by the ansatz[24]

$$\Phi_k(z) = k\phi_k(z) + g_k(z) \tag{10.4}$$

Substitution in (9.8) and use of the fact that ϕ_k is a solution yield after some algebra the following inhomogeneous equation for $g_k(z)$:

$$g_{k+2} - 2g_{k+1} + g_k = z^{k+m} g_{k+l} + (l-2)\,\phi_{k+2} - 2(l-1)\,\phi_{k+1} + l\phi_k \tag{10.5}$$

This equation can be solved by assuming

$$g_k = \sum_{n=0}^{\infty} \frac{z^{n[l(n-1)+2(k+m)]/2}}{q_n^2} p_n(z) \tag{10.6}$$

Indeed, after a long but straightforward calculation, one concludes that p_n must satisfy

$$p_{n+1}(z) = p_n(z) + l - 2 + \frac{2}{1 - z^{n+1}} \qquad \text{for} \quad n \geqslant 0 \tag{10.7}$$

Note that $\Phi_k(z)$ can be redefined up to an additive term of the form $h(z)\,\phi_k(z)$. This allows the convenient choice $p_0(z) \equiv 0$, yielding

$$p_n(z) = n(l-2) + s_n(z) \qquad \text{for} \quad n \geqslant 1 \tag{10.8}$$

where $s_n(z)$ is defined by (6.17). Finally, the second "physical" solution of (9.8) is obtained as

$$\Phi_k(z) = k + \sum_{n=1}^{\infty} \frac{z^{n[l(n-1)+2(k+m)]/2}}{q_n^2} [k + n(l-2) + s_n] \tag{10.9}$$

Note that this choice corresponds to $\rho_\infty(z) \equiv 1$ in (10.3).

11. SUMMARY AND DISCUSSION

Our presentation of the solutions for several cluster statistics models illustrates the important features of this class of *compact* animals. If the stacking rules are sufficiently relaxed to allow for clusters with finite entropy $N^{-1} \ln c_n \to \ln \lambda$ per element (circle, square) in the large-N limit, i.e., $\lambda > 1$, then the universal form (2.8) applies with $\theta = 0$. Models with more restrictive rules have entropy vanishing as $N^{-1/2}$ [see (6.10), (6.11), (7.7)].

All the models studied were two-dimensional. In three dimensions, only limited results are available, for two cases. The first model generalizes the square stackings in a corner[2,6] to stackings of cubes in a $3d$ corner.[11,2] The second model is a certain $3d$ version[10] of the partially directed compact animals. It is hoped that more progress will be possible for $d > 2$ models in the future.

Results on the *average sizes* and *shape* of compact clusters have been derived for square stackings in a corner[2,6] and for the partially directed compact animals.[10] Much work remains to be done on this issue.

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